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Connection between conserved quantities of the Hamiltonian and of the S-matrix

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Abstract. A conserved quantity K of a Hamiltonian H implies a conserved quantity J of the S-matrix only if the forward and the backward asymptotic limits of K are equal. A conserved quantity J of S implies a conserved quantity of H only if J is also a conserved quantity of the free Hamiltonian H_0. Therefore the integrability or chaoticity of S does not necessarily decide the integrability of H.

1. Introduction

In recent years there has been a growing interest in the integrability and chaos of scattering systems (for reviews see [1, 2]). The classical case has been analysed quite thoroughly and the essential result is the following. When there is topological chaos in the phase space, there exists a chaotic saddle consisting of unstable localized orbits. Thereby the deflection function and the time delay function become singular on a fractal subset of their domain. That is whenever the initial condition of a scattering trajectory lies on the stable manifold of some localized orbit, this trajectory remains in the interaction region and the time delay is infinite and the scattering angle is undefined. Of course this only occurs for a subset of measure zero in the set of all incoming asymptotes. We can interpret scattering chaos as the Hamiltonian version of transient chaos [3–5].

Most scattering experiments are done on micro systems, where quantum effects are essential, so we need to know how scattering chaos manifests itself in the quantum world. As there are no individual particle trajectories in quantum dynamics, it is impossible to investigate the deflection function and the time delay function for the structure of their singularities. We have to look for more indirect signs of chaos. Since the S-matrix is the central object in quantum scattering theory, investigating the statistical properties of the S-matrix and interpreting random matrix behaviour according to the orthogonal ensemble of random unitary matrices as signs of chaos was proposed [2]. Unfortunately, it became clear [6] that the random matrix behaviour of the eigenphases of S is not necessarily related to topological chaos in the corresponding classical system. It has been shown, by semiclassical considerations [7], that the behaviour of the phases of S is linked to the behaviour of the classical iterated scattering map $\tilde{M}$ as constructed in [8–10] and this is explained in more detail below. The eigenphase statistics of S shows random matrix behaviour according to the COE statistics whenever the classical map $\tilde{M}$ is dominated by unstable periodic points. In addition chaos in $\tilde{M}$ is not necessarily linked to the topological chaos of the flow in the phase space generated by H, where H is the classical Hamiltonian of the system. So the question arises: what kind of chaos and integrability do the quantities $\tilde{M}$ and S really test for. The integrability of $\tilde{M}$ will be understood in the same spirit as that of a classical conservative map, and the integrability of S as that of a Hamiltonian operator.

To shed some light on these problems, it would be useful to have a better understanding of the connection between the integrability and chaos of S and $\tilde{M}$, on the one hand, and of H, on the other hand. Therefore we address the following questions in this paper.

(i) Under which additional conditions does the integrability of S or $\tilde{M}$ imply the integrability of H?

(ii) Under which additional conditions does the integrability of H imply the integrability of S and $\tilde{M}$?

In section 2 the meaning of integrability of S and $\tilde{M}$ is considered. Section 3 gives the conditions for the transfer of integrability between H and S or $\tilde{M}$. Section 4 gives some illustrative examples. In order to give an explanation which is as complete as possible, we perform all considerations classically and quantum mechanically in parallel.

2. Integrability of S and M

For simplicity in sections 2 to 4 let us consider the motion of a point particle in a two-dimensional position space under the influence of a time-independent, local, short-range potential. We assume that the asymptotic conditions and the asymptotic completeness are always fulfilled.

Let q be the position and p the momentum of the particle. The classical dynamics is generated by the Hamiltonian function

$$H(q,p) = H_0(p) + V(q) \tag{1}$$

where $H_0(p) = p^2/2m$ is the free Hamiltonian. The quantum dynamics is generated by the corresponding operator $\hat{H}$. Let $\Phi(T)$ be the flow map in classical phase space generated by H. It shifts each point along its trajectory by the curve parameter T (= run time). Let $\Phi_0(T)$ be the flow map generated by H_0. For the classical considerations it is useful to introduce the coordinates p, α, b, u in phase space. p is the absolute value of p. $\alpha = \tan^{-1}(p_y/p_x)$ is the direction of p. b is the component of q perpendicular to p. u is the component of q parallel to p. In these coordinates $\Phi_0(T)$ has the simple form

$$\Phi_0(T)(p, \alpha, b, u) = (p, \alpha, b, u + Tp/m). \tag{2}$$

Asymptotes are trajectories of Φ_0 and along them p, α, b stay constant. Accordingly we can label asymptotes by giving a set of values of p, α, b.

H is integrable if there exists a function K on phase space which is independent of H and fulfils

$$\{H, K\} = 0. \tag{3}$$

This condition implies that $K \circ \Phi(T) = K$ for any $T \in \mathbb{R}$. The corresponding quantum condition is the existence of an operator $\hat{K}$ independent of $\hat{H}$ such that

$$[\hat{H}, \hat{K}] = 0. \tag{4}$$

The S-matrix is defined as

$$\hat{S} = \hat{\Omega}_-^\dagger \hat{\Omega}_+ \tag{5}$$

where the Moller operators are given by

$$\hat{\Omega}_\pm = \lim_{T \to \mp\infty} \exp(-\mathrm{i}\hat{H}T/\hbar) \exp(\mathrm{i}\hat{H}_0 T/\hbar). \tag{6}$$

The classical analogue of $\hat{S}$ is the scattering map M defined by

$$M = \lim_{T, T' \to \infty} \Phi_0(-T) \circ \Phi(T + T') \circ \Phi_0(-T'). \tag{7}$$

The map M takes any point x in phase space and performs a three-step operation on it. First we construct the trajectory of Φ_0 through x and follow it backwards in time into the incoming asymptotic region. There we switch to the dynamics Φ generated by the full Hamiltonian H and let the point run forward in time through a full scattering trajectory until it reaches the outgoing asymptotic region. In the last step we switch again to the dynamics Φ_0 generated by H_0 and let time run backwards again. A detailed explanation of this construction is given in [11] and in section 3.4 of [12].

We shall call M integrable if there is a function J defined on phase space which is independent of H_0, such that

$$J \circ M = J. \tag{8}$$

Then the phase space is foliated into level surfaces of J and the map M transports any point of some level surface into a point lying on the same level surface.

The corresponding quantum condition is the existence of an operator $\hat{J}$, independent of $\hat{H}_0$, fulfilling

$$[\hat{J}, \hat{S}] = 0. \tag{9}$$

The significance of equation (9) lies in the following. There is a set of eigenfunctions in which $\hat{J}$ and $\hat{S}$ are both diagonal, i.e. we have a basis in which $\hat{S}$ is diagonal independent of the value of the energy. In this case the matrix elements of $\hat{S}$, even when averaged over the energy, can never fulfil the random matrix properties as described in [2]. In addition, the eigenphases of $\hat{S}$ may cross when some parameter (e.g. the energy) is varied and their distribution will violate the properties we expect for COE statistics. So equations (8) and (9) imply the non-chaoticity of M or S.

The map constructed in equation (7) is not exactly the one constructed in [8–10] and used in the semiclassical considerations in [7]. There, a map $\tilde{M}$ has been constructed by which incoming asymptotes are mapped into incoming asymptotes by the following prescription. Take any incoming asymptote labelled by (p, α, b). Follow the corresponding scattering trajectory until it ends in an outgoing asymptote labelled by (p', α', b'). Now identify this outgoing asymptote with the incoming asymptote having the same values of (p', α', b'). In total we have a map $\tilde{M}$ given by

$$\tilde{M}(p, \alpha, b) = (p', \alpha', b'). \tag{10}$$

We may interpret $\tilde{M}$ as a simplified version of M where we disregard the value of T and T' in the argument of Φ_0 in equation (7). The time parameter in the free motion generated by H_0 is irrelevant for all questions concerning the properties of initial and final asymptotes as a whole trajectory only, i.e. for properties which are not sensitive to the position along asymptotes.

For the integrability of $\tilde{M}$ it is relevant whether there exists a function $\tilde{J}$, defined on asymptotes, which fulfils

$$\tilde{J} \circ \tilde{M} = \tilde{J}. \tag{11}$$

Now it is important how integrability of M is connected with integrability of $\tilde{M}$. $\tilde{J}$ is defined on asymptotes as a whole not on individual points of phase space. Therefore $\tilde{J}$ is a function of p, α, b only, it does not depend on u. So we automatically have that

$$\tilde{J} \circ \Phi_0 = \tilde{J}. \tag{12}$$

Then the factors Φ_0 in equation (7) are irrelevant for the question of invariance of $\tilde{J}$ and equation (11) implies

$$\tilde{J} \circ M = \tilde{J}. \tag{13}$$

On the other hand a conserved quantity J of M is a conserved quantity of $\tilde{M}$ only if J fulfils the additional condition

$$J \circ \Phi_0 = J. \tag{14}$$

For the semiclassical considerations in [7] the integrability of $\tilde{M}$ is important, i.e. the existence of such conserved quantities of M which are invariant under Φ_0 in addition. Therefore we will direct our attention to functions J which fulfil equation (8) and in addition the condition (14).

Some of the classical objects we have already constructed and will construct in the following do not exist on the complete phase space but only on points lying on generic scattering trajectories having proper in and out asymptotes. Let us denote this part of the phase space by $\mathcal{P}$. In particular we have to leave out the localized orbits and their invariant manifolds. If the flow Φ is not chaotic, then this omission does not cause any problems. So we restrict our considerations to systems without topological chaos in the energy surfaces to positive values of the energy. The other systems, the ones displaying scattering chaos, are not interesting in the present context, because for such systems neither H nor M can ever be integrable.

In addition some of the quantum objects we have to deal with are defined on scattering states only. Let us denote by $\mathcal{R}$ the subspace of the Hilbert space $\mathcal{H}$ which consists of scattering states only.

3. Transfer of integrability between S and H

In this section we consider the connection between integrability of M or $\hat{S}$ and the integrability of H. It is more convenient to work first in the direction from integrability of M or $\hat{S}$ to the integrability of H, even though the other direction is the more important one for applications.

Let us begin with the classical statement formulated as follows.

Proposition 1C. Let there be given a function J fulfilling equation (8). If J fulfils equation (14) in addition, then the function

$$K = \lim_{T\to\infty} J \circ \Phi_0(-T) \circ \Phi(T) \tag{15}$$

whose domain is $\mathcal{P}$, fulfils the two conditions:

(a) K is a conserved quantity of H;

(b) the forward and backward asymptotic limits of K are the same, i.e. K fulfils the equation

$$\lim_{T\to\infty} K \circ \Phi(T) \circ \Phi_0(-T) = \lim_{T\to\infty} K \circ \Phi(-T) \circ \Phi_0(T). \tag{16}$$

Proof of (a).

$$K \circ \Phi(t) = \lim_{T\to\infty} J \circ \Phi_0(T) \circ \Phi(-T) \circ \Phi(t).$$

First we use the property $\Phi(a) \circ \Phi(b) = \Phi(a+b)$ to transform this expression into the form

$$\lim_{T\to\infty} J \circ \Phi_0(T) \circ \Phi(-T+t).$$

Next set $T = T' + t$ giving

$$\lim_{T'\to\infty} J \circ \Phi_0(T'+t) \circ \Phi(-T') = \lim_{T'\to\infty} J \circ \Phi_0(t) \circ \Phi_0(T') \circ \Phi(-T').$$

Finally we use the invariance of J under Φ_0, i.e. equation (14), to absorb the factor $\Phi_0(t)$ into J and obtain the final expression

$$\lim_{T'\to\infty} J \circ \Phi_0(T') \circ \Phi(-T') = K.$$

Proof of (b).

$$\begin{aligned}\lim_{T\to\infty} K \circ \Phi(T) \circ \Phi_0(-T) &= \lim_{T\to\infty} \lim_{T'\to\infty} J \circ \Phi_0(-T') \circ \Phi(T') \circ \Phi(T) \circ \Phi_0(-T) \\ &= J \circ M = J.\end{aligned}$$

Here we insert identity maps in the form id $= \Phi_0(-T) \circ \Phi_0(T)$ and id $= \Phi(T) \circ \Phi(-T)$ to obtain

$$\begin{aligned}\lim_{T\to\infty} J \circ \Phi_0(-T) \circ \Phi_0(T) &= \lim_{T\to\infty} J \circ \Phi_0(-T) \circ \Phi(T) \circ \Phi(-T) \circ \Phi_0(T) \\ &= \lim_{T\to\infty} K \circ \Phi(-T) \circ \Phi_0(T).\end{aligned}$$

The corresponding quantum result is as follows.

Proposition 1Q. Let there be given an operator $\hat{J}$ fulfilling equation (9). If $\hat{J}$ fulfils $[\hat{J}, \hat{H}_0] = 0$ in addition, then the operator

$$\hat{K} = \hat{\Omega}_+ \hat{J} \hat{\Omega}_+^\dagger \tag{17}$$

whose domain is $\mathcal{R}$ fulfils the two conditions:

(a) $\hat{K}$ is a conserved quantity of $\hat{H}$;

(b) the forward and the backward asymptotic limits of $\hat{K}$ are equal, i.e. $\hat{K}$ fulfils the equation

$$\hat{\Omega}_-^\dagger \hat{K} \hat{\Omega}_- = \hat{\Omega}_+^\dagger \hat{K} \hat{\Omega}_+ . \tag{18}$$

In general we cannot make any statement about an extension of $\hat{K}$ to an operator defined on the whole of $\mathcal{H}$.

Proof of (a). Here we use the intertwining relations [13, 14]

$$\hat{H} \hat{\Omega}_\pm = \hat{\Omega}_\pm \hat{H}_0 \tag{19}$$

and their adjoints $\hat{\Omega}_\pm^\dagger \hat{H} = \hat{H}_0 \hat{\Omega}_\pm^\dagger$ where the Hamiltonian operators $\hat{H}$ and $\hat{H}_0$ are assumed to be self-adjoint and obtain

$$\hat{K} \hat{H} = \hat{\Omega}_+ \hat{J} \hat{\Omega}_+^\dagger \hat{H} = \hat{\Omega}_+ \hat{J} \hat{H}_0 \hat{\Omega}_+^\dagger .$$

Because of our assumption on $\hat{J}$ we have $\hat{J} \hat{H}_0 = \hat{H}_0 \hat{J}$ giving

$$\hat{\Omega}_+ \hat{H}_0 \hat{J} \hat{\Omega}_+^\dagger = \hat{H} \hat{\Omega}_+ \hat{J} \hat{\Omega}_+^\dagger = \hat{H} \hat{K} .$$

Proof of (b).

$$\hat{\Omega}_+^\dagger \hat{K} \hat{\Omega}_+ = \hat{\Omega}_+^\dagger \hat{\Omega}_+ \hat{J} \hat{\Omega}_+^\dagger \hat{\Omega}_+ .$$

Now we use $\hat{\Omega}_\pm^\dagger \hat{\Omega}_\pm = \mathbb{1}$ to see that this expression is just $\hat{J}$. Next we use our assumption that equation (9) is fulfilled to obtain

$$\hat{J} = \hat{S} \hat{J} \hat{S}^\dagger = \hat{\Omega}_-^\dagger \hat{\Omega}_+ \hat{J} \hat{\Omega}_+^\dagger \hat{\Omega}_- = \hat{\Omega}_-^\dagger \hat{K} \hat{\Omega}_- .$$

Remark. In order to construct $\hat{K}$ in equation (17) $\hat{J}$ is transformed by $\hat{\Omega}_+$ apparently preferring one direction in time. However, from (b) of the proposition and its proof it becomes evident that $\hat{\Omega}_- \hat{J} \hat{\Omega}_-^\dagger = \hat{\Omega}_+ \hat{J} \hat{\Omega}_+^\dagger$. Therefore we could have taken the definition $\hat{K} = \hat{\Omega}_- \hat{J} \hat{\Omega}_-^\dagger$ instead of equation (17) to obtain the same operator $\hat{K}$. In this way the symmetry between future and past is maintained.

Next we come to the reverse direction. The classical result is formulated as follows.

Proposition 2C. Let there be a function K fulfilling equation (3). We define the functions J_+, J_- by

$$J_+ = \lim_{T\to\infty} K \circ \Phi(-T) \circ \Phi_0(T) \tag{20}$$

$$J_- = \lim_{T\to\infty} K \circ \Phi(T) \circ \Phi_0(-T). \tag{21}$$

We call J_+, J_- the forward and backward asymptotic limits of K. If these two limits are equal, i.e. if

$$J_+ = J_- \tag{22}$$

then we define the function $J = J_+ = J_-$ which fulfils the two properties
(a) J is a conserved quantity of M;
(b) J is a conserved quantity of H_0.

Proof of (a).

$$\begin{aligned} J \circ M &= \lim_{T\to\infty} \lim_{T'\to\infty} \lim_{T''\to\infty} K \circ \Phi(-T) \circ \Phi_0(T) \circ \Phi_0(-T') \circ \Phi(T') \circ \Phi(T'') \circ \Phi_0(-T'') \\ &= \lim_{T''\to\infty} K \circ \Phi(T'') \circ \Phi_0(-T'') = J. \end{aligned}$$

Proof of (b).

$$\begin{aligned} J \circ \Phi_0(t) &= \lim_{T\to\infty} K \circ \Phi(-T) \circ \Phi_0(T) \circ \Phi_0(t) \\ &= \lim_{T\to\infty} K \circ \Phi(-T) \circ \Phi_0(T+t) = \lim_{T'\to\infty} K \circ \Phi(-T'+t) \circ \Phi_0(T') \\ &= \lim_{T'\to\infty} K \circ \Phi(t) \circ \Phi(-T') \circ \Phi_0(T') = \lim_{T'\to\infty} K \circ \Phi(-T') \circ \Phi_0(T') = J. \end{aligned}$$

The corresponding quantum statement is

Proposition 2Q. Let there be an operator $\hat{K}$ fulfilling equation (4). We define the operators $\hat{J}_+, \hat{J}_-$ as

$$\hat{J}_+ = \hat{\Omega}_-^\dagger \hat{K} \hat{\Omega}_- \tag{23}$$

$$\hat{J}_- = \hat{\Omega}_+^\dagger \hat{K} \hat{\Omega}_+. \tag{24}$$

We call $\hat{J}_+, \hat{J}_-$ the forward and backward asymptotic limits of $\hat{K}$. If these two limits are equal, i.e. if

$$\hat{J}_+ = \hat{J}_- \tag{25}$$

then we define $\hat{J} = \hat{J}_+ = \hat{J}_-$ which fulfils the two properties:
(a) $\hat{J}$ is a conserved quantity of $\hat{S}$;
(b) $\hat{J}$ is a conserved quantity of $\hat{H}_0$.

Remark. Before we prove these statements let us make a remark about the definitions (23) and (24). $\hat{\Omega}_\pm$ is defined on the whole Hilbert space $\mathcal{H}$. Its range is $\mathcal{R}$. Because of equation (4) the operator $\hat{K}$ conserves the energy, i.e. $\hat{K}$ applied to any scattering state again gives a pure scattering state and the result does not contain any bound state parts. Therefore $\hat{\Omega}_\pm^\dagger$ can be applied to the range of $\hat{K}\hat{\Omega}_\pm$ and the operators $\hat{J}_+$ and $\hat{J}_-$ are well defined on the whole of $\mathcal{H}$. Similar considerations will hold in several other places in the following.

Proof of (a).

$$\hat{S}\hat{J} = \hat{\Omega}_-^\dagger\hat{\Omega}_+\hat{\Omega}_+^\dagger\hat{K}\hat{\Omega}_+ = \hat{\Omega}_-^\dagger\hat{K}\hat{\Omega}_+ = \hat{\Omega}_-^\dagger\hat{K}\hat{\Omega}_-\hat{\Omega}_-^\dagger\hat{\Omega}_+ = \hat{J}\hat{S}.$$

Note that applied on vectors from $\mathcal{R}$ the product $\hat{\Omega}_\pm\hat{\Omega}_\pm^\dagger$ acts like the unit operator even though $\hat{\Omega}_\pm$ are not unitary operators on the whole of $\mathcal{H}$.

Proof of (b).

$$\hat{J}\hat{H}_0 = \hat{\Omega}_-^\dagger\hat{K}\hat{\Omega}_-\hat{H}_0 = \hat{\Omega}_-^\dagger\hat{K}\hat{H}\hat{\Omega}_- = \hat{\Omega}_-^\dagger\hat{H}\hat{K}\hat{\Omega}_- = \hat{H}_0\hat{\Omega}_-^\dagger\hat{K}\hat{\Omega}_- = \hat{H}_0\hat{J}.$$

Now we have finished the proof of the equivalence of the following two statements A and B.

(A) H has a conserved quantity with equal forward and backward asymptotic limits.

(B) M or S has a conserved quantity which is also a conserved quantity of H_0.

In passing let us mention a further observation which is not essential for our main problem but which may be interesting in itself. In the proof of part (a) of proposition 2Q the condition (4) was never really used. Therefore the following holds true. Whenever we find an operator $\hat{K}$ with $\hat{K}\mathcal{R} \subset \mathcal{R}$ and which fulfils equation (18), then the operator $\hat{J} = \hat{\Omega}_-^\dagger\hat{K}\hat{\Omega}_- = \hat{\Omega}_+^\dagger\hat{K}\hat{\Omega}_+$ commutes with $\hat{S}$.

4. Examples

(i) From the well known examples of symmetries of the S-matrix as presented in the text books [13, 14], we are familiar with the following situation. There is a conserved quantity $\hat{K}$ fulfilling equation (4) and this $\hat{K}$ is a conserved quantity of the free motion in addition, i.e. $\hat{K}$ fulfils $[\hat{K}, \hat{H}_0] = 0$. Then $\hat{J}_+ = \hat{J}_- = \hat{K}$ and the condition (25) is fulfilled trivially. Examples of this type are the angular momentum for rotationally symmetric potential, the parity, time reversal etc.

(ii) Now we present an example where $\{H_0, K\} \neq 0$ but condition (22) is satisfied. Choose $V = (\cos\phi)^2/2r^2$ where r, ϕ are polar coordinates in position space. A conserved quantity of H is $K = L^2 + (\cos\phi)^2$ where L is the angular momentum. Direct computation gives $\{H, K\} = 0$, $\{H_0, K\} = 2L\cos\phi\sin\phi/r^2 \neq 0$. To compute the asymptotic limits of K consider that far away from the origin in the asymptotic region we have $\phi \to \alpha$ along outgoing asymptotes and $\phi \to \alpha - \pi$ along incoming asymptotes. Therefore $\cos\phi = x/r \to p_x/p$ along outgoing asymptotes and $\cos\phi \to -p_x/p$ along incoming asymptotes. Therefore

$$J = J_+ = J_- = L^2 + p_x^2/p^2.$$

Considering that L and p are conserved under the free flow Φ_0 it is evident that $J \circ \Phi_0 = J$. In a numerical plot of the iterated scattering map $\tilde{M}$ the asymptotic α/b-plane is foliated into invariant level lines of J.

Quantum mechanically the corresponding results are obtained as follows.

$$\hat{K} = \hat{L}^2 + \hat{x}^2/(\hat{x}^2 + \hat{y}^2).$$

In this expression no problems of ordering occur. Because of $[\hat{K}, \hat{H}] = 0$ we find

$$\exp(\mathrm{i}\hat{H}T/\hbar)\hat{K}\exp(-\mathrm{i}\hat{H}T/\hbar) = \hat{K} \qquad \text{for any } T \in \mathbb{R}.$$

Next we use the well known relations

$$\exp(-\mathrm{i}\hat{H}_0 T/\hbar)\hat{q}\exp(\mathrm{i}\hat{H}_0 T/\hbar) = \hat{q} - \hat{p}\,T/m$$

$$\exp(-\mathrm{i}\hat{H}_0 T/\hbar)\hat{p}\exp(\mathrm{i}\hat{H}_0 T/\hbar) = \hat{p}$$

$$\exp(-\mathrm{i}\hat{H}_0 T/\hbar)\hat{L}\exp(\mathrm{i}\hat{H}_0 T/\hbar) = \hat{L}$$

to obtain

$$\begin{aligned}\exp(-\mathrm{i}\hat{H}_0 T/\hbar)&\exp(\mathrm{i}\hat{H}T/\hbar)\hat{K}\exp(-\mathrm{i}\hat{H}T/\hbar)\exp(\mathrm{i}\hat{H}_0 T/\hbar)\\ &= \hat{L}^2 + (\hat{x} - \hat{p}_x T/m)^2/[(\hat{x} - \hat{p}_x T/m)^2 + (\hat{y} - \hat{p}_y T/m)^2].\end{aligned}$$

In the limit $T \to \pm\infty$ this expression becomes

$$\hat{\Omega}_\pm^\dagger \hat{K}\hat{\Omega}_\pm = \hat{L}^2 + \hat{p}_x^2/(\hat{p}_x^2 + \hat{p}_y^2).$$

In the final expression there are no ordering problems even though they occur in some intermediate steps. Fortunately they are not important for our present considerations.

The integrability of the scattering process represented by M or S can be understood as follows. The foliation of the phase space into level sets of J is invariant under the flow Φ_0, i.e. any trajectory of H_0 stays in a level set of J. In the same way the foliation of the phase space into level sets of K is invariant under the flow Φ, i.e. any trajectory of H stays in a level set of K. The essential property of J and K is that in the asymptotic region the numerical values of J and K coincide. Thereby the foliations generated by J and K also coincide in this region. Any scattering trajectory lies in a level set of K but in general it does not lie in a level set of J. In the interaction region the value of J is not constant along the scattering trajectories. However, and this is the important point, in the outgoing asymptotic region any scattering trajectory comes back to exactly the same numerical value of J at which it has started in the incoming asymptotic region, regardless of the values of J in between. This is guaranteed by the property that J is the forward as well as the backward asymptotic limit of K. Therefore the three-step mapping process described by M maps any point of phase space to a point lying on the same level set of J.

(iii) A very illuminating example is provided by the energy. $\hat{H}$ is certainly a conserved quantity under the motion generated by $\hat{H}$, but in general $[\hat{H}, \hat{H}_0] \neq 0$. If we set $\hat{K} = \hat{H}$ then the condition (25) is always fulfilled because the intertwining relations (19) guarantee that $\hat{J}_+ = \hat{J}_- = \hat{H}_0$. Therefore $\hat{H}_0$ is a conserved quantity of $\hat{S}$, a well known result [13, 14].

In analogy to the situation for the energy we may reformulate the conditions (17) and (25) as follows. There are operators $\hat{J}$ and $\hat{K}$ fulfilling the two equations

$$\hat{\Omega}_+ \hat{J} = \hat{K}\hat{\Omega}_+ \tag{26}$$

$$\hat{\Omega}_- \hat{J} = \hat{K}\hat{\Omega}_- . \tag{27}$$

We call these equations generalized intertwining relations and call the operators $\hat{K}$ and $\hat{J}$ related by an intertwining relation.

(iv) Now we provide an example, where condition (22) or (25) is not fulfilled. Choose $V = (\cos\phi + c)/2r^2$ where the constant c should be taken as $c > 1$ in order to avoid an attractive singularity of the potential which would create unnecessary complications. Direct computation shows that $K = L^2 + \cos\phi$ is a conserved quantity of the full motion. Along the same pattern as in example (ii) we obtain the asymptotic limits of K as $J_+ = L^2 + p_x/p$, $J_- = L^2 - p_x/p$. Therefore K does not imply integrability of M or S.

Could there be another conserved quantity K' of H which provides a related conserved quantity J' of M or S? Numerical evidence speaks against this possibility. Because the energy is a conserved quantity in any Hamiltonian system we can restrict the considerations to one particular value of E. For fixed E the set of all asymptotes is a two-dimensional manifold, and for its coordinates we use b and α. Figure 1 gives the iterated application of $\tilde{M}$ to several initial points (marked by crosses) in the α/b plane. Clearly we see chaotic behaviour in a strip of the plane. The iterated points of one initial point are not restricted to the level line of any function J. Because the potential is homogeneous of degree minus two in the position coordinates, the plot of $\tilde{M}$ is independent of the value of E.

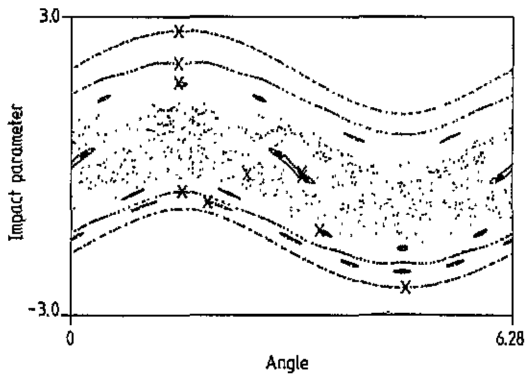

Figure 1. Iterated scattering map for example system (iv) and parameter value $c = 2$. A few hundred iterates of some initial points (marked by crosses) are plotted.

For the quantum system in figure 2 we give a plot of some eigenphases of the symmetric part of $\hat{S}$ as a function of the potential parameter c. In the quantum case the parameter c can start at $c = 0.93$, since the lowest eigenvalue of $\hat{K}$ is larger than -0.93. We see avoided crossings giving a numerical indication that $\hat{S}$ is not integrable. $\hat{S}$ is not diagonal in a representation in eigenfunctions of $\hat{K}$.

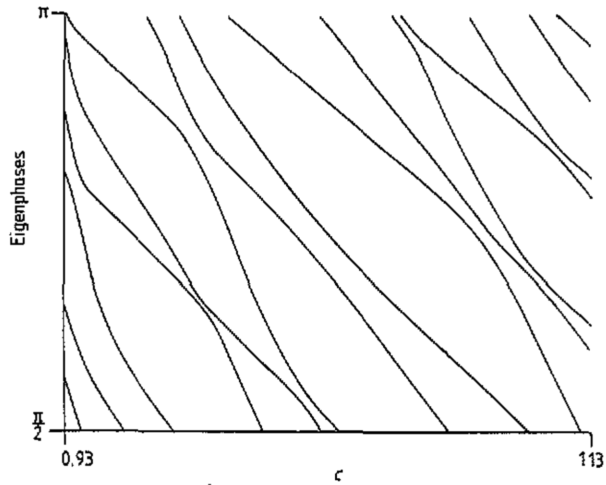

Figure 2. Plot of some eigenphases of the symmetric part of the S-matrix as a function of the potential parameter c for example system (iv).

Therefore we have found a completely integrable system providing a chaotic scattering map and S-matrix. Some more systems of this type are presented in [6].

(v) Finally we give an example of a function J which is invariant under M but which does not fulfil the additional condition (14) and which is therefore not an invariant of $\tilde{M}$.

Take any system with rotationally symmetric potential. As before, asymptotes are labelled by p, α, b. Following scattering trajectories from ingoing to outgoing asymptotes we get the mapping $\tilde{M} : (p, \alpha, b) \to (p', \alpha', b')$. Because of conservation of energy $p' = p$ and because of conservation of angular momentum $b' = b$. $\alpha' = \alpha + \theta(p, b)$ where $\theta(p, b)$ is the scattering angle. The map M acts like

$$M(p, \alpha, b, u) = (p, \alpha + \theta(p, b), b, u - \Delta(p, b)). \tag{28}$$

Here $\Delta(p, b) = pDt(p, b)/m$ where $Dt(p, b)$ is the time delay of a scattering trajectory with these values of p and b. Because of the rotational symmetry of the system there is no topological chaos in this system. Therefore Dt is not a chaotic function and Δ is at least piecewise smooth. So we can define a function J by

$$J(p, \alpha, b, u) = \sin[2\pi u/\Delta(p, b)]g(\Delta(p, b)). \tag{29}$$

For $g(z)$ we can take any function which goes to zero sufficiently fast, when its argument goes to zero. The function g is included in order to avoid trouble at places where $\Delta \to 0$. This always happens when $b \to \infty$. Because of the periodicity of J with period Δ in its argument u we obtain using (28)

$$J \circ M(p, \alpha, b, u) = J(p, \alpha + \theta(p, b), b, u - \Delta(p, b)) = J(p, \alpha, b, u)$$

for any $(p, \alpha, b, u) \in \mathcal{P}$, i.e. J is an invariant of M. However,

$$J(p, \alpha, b, u) \neq J(p, \alpha, b, u + d)$$

for most $d \in \mathbb{R}$. Therefore $J \circ \Phi_0 \neq J$ and condition (14) is not satisfied. Obviously a function K constructed from J by equation (15) would not be constant along trajectories of Φ either.

Of course, by (29) we have made a strange choice for J. If we had taken $J' = b$, then J' would fulfil (14) and the corresponding K' would be essentially the angular momentum, which is a good conserved quantity of H. This example shows that condition (14) is essential in order to obtain a useful conserved quantity of M or S.

5. Arbitrary number of degrees of freedom

So far all results have been given for the case of two degrees of freedom. Finally we give the corresponding statements for the case of n degrees of freedom.

The following two conditions are equivalent.

(i) The system is completely integrable, i.e. there are n independent functions K_i on phase space or operators $\hat{K}_i$ where H or $\hat{H}$ is one of them. They fulfil $\{K_i, K_j\} = 0$ or $[\hat{K}_i, \hat{K}_j] = 0$ for all i, j. In addition, for each of these K_i or $\hat{K}_i$ the forward and backward asymptotic limits are equal.

(ii) There is a set of n independent functions J_i or operators $\hat{J}_i$ (they are the asymptotic limits of the K_i from condition (i)) which are conserved quantities of M or $\hat{S}$, i.e. $J_i \circ M = J_i$ or $[\hat{J}_i, \hat{S}] = 0$ for all i. H_0 or $\hat{H}_0$ is one of them. In addition, they fulfil $\{J_i, J_j\} = 0$ or $[\hat{J}_i, \hat{J}_j] = 0$ for all i, j. Because H_0 is one of the J_i, all of them are constants of the free motion.

The proofs of these statements are verbatim repetitions of the proofs we have given for the case of two degrees of freedom. The whole sets of $\hat{K}_i$ and $\hat{J}_i$ are related by the intertwining relations $\hat{\Omega}_{\pm} \hat{J}_i = \hat{K}_i \hat{\Omega}_{\pm}$ for all i.

6. Final remarks

The main result of all our considerations is the following. A conserved quantity K of the Hamiltonian implies a conserved quantity J of the S-matrix only if K and J are related by a generalized intertwining relation. This is the case only if the forward and backward asymptotic limits of K are equal. Therefore a check of integrability of S is not a sufficient test for integrability of H. It only tests for a restricted type of integrability. As example (iv) in section 4 and some more examples of this kind and there quantum counterparts in [6] demonstrate, an integrable Hamiltonian can lead to a chaotic map M.

As has been shown in [6], a good scattering test for integrability of H is an investigation of the energy dependence of the cross section. Ericson fluctuations caused by very many overlapping resonances are a good indication of classical chaos. These results are not surprising in view of the following. The properties of $\hat{S}$ and, in particular, the scattering phases are created by a delicate interplay between the full motion generated by H and the free motion generated by H_0. In most cases it is hard to disentangle any properties of S into the contributions reflecting properties of H and the ones reflecting properties of H_0. However, we usually choose a H_0 so that it does not produce resonances, e.g. when we choose the kinetic energy for H_0. Then all resonances are only caused by the structure of H. Accordingly, the distribution of resonances is one of the few properties of S where the disentanglement between effects coming from H and H_0 is trivial. It might be an interesting task to investigate the resonance structure of $\hat{S}$ in a system, where H and H_0 both create resonances.

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